$$\text{MSC} = \begin{cases} \mathbf{11-02} \\ \mathbf{11R52} \end{cases}$$

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$\mathfrak{z}$

The Spacetime Rotations of the Quaternion

Q8 Group

$$\mathrm{i} \quad -\mathrm{i} \quad \mathrm{j} \quad -\mathrm{j} \quad \mathrm{k} \quad -\mathrm{k} \quad 1 \quad -1$$

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

$$\mathrm{ij} = \mathrm{k}, \ \mathrm{jk} = \mathrm{i}, \ \mathrm{ki} = \mathrm{j}, \ \mathrm{ji} = -\mathrm{k}, \ \mathrm{kj} = -\mathrm{i}, \ \mathrm{ik} = -\mathrm{j}$$

$\mathrm{q} = \varkappa + \mathrm{x}i + \mathrm{y}j + \mathrm{z}k$ **, where** $\varkappa, \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{R}$

SPACETIME QUATERNION

Let us assume the Quaternion as a vector space where in addition to the 3 Space Dimensions $\mathrm{x,y,z}$ **over the IM axes** i,j,k **are a vector space Velocity** ς **and a Time dimension** t **both measurable.**

$\varsigma \geq 0$ **is over all the axes of the quaternion**

t **is the Time into the quaternion**

σ **is the Space dimension given by** $\mathrm{x,y,z}$ **over the IM axes** i,j,k

Assuming $\varkappa = t\varsigma$

$\mathfrak{z} = t\varsigma + \mathrm{x}i + \mathrm{y}j + \mathrm{z}k$ **is the SpaceTime quaternion**

$$\sigma = t\varsigma = \mathrm{x}i + \mathrm{y}j + \mathrm{z}k$$

$$\mathfrak{z} = 2\varkappa = 2t\varsigma = 2(\mathrm{x}i + \mathrm{y}j + \mathrm{z}k) = 2\sigma$$

$\sigma = t\varsigma$ **thence** $\mathrm{Re} = \mathrm{Im}$

$$\|\mathrm{t}\| = \sqrt{\frac{-(\mathrm{x}i)^2 - (\mathrm{y}j)^2 - (\mathrm{z}k)^2}{\varsigma^2}}$$

, using the signature (1,3) and the Speed vector as a Scalar, not depending on the orientation

in simplified form

$$t = \frac{\sqrt{\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2}}{\varsigma}$$

Now, we know the length of t **and its coordinates** $\frac{\mathrm{x}}{\varsigma}, \frac{\mathrm{y}}{\varsigma}, \frac{\mathrm{z}}{\varsigma}$ **over their versors** i,j,k **and its direction by the vectors** $\vec{0\mathrm{x}}, \vec{0\mathrm{y}}, \vec{0\mathrm{z}}$

VECTOR SPACE

$$\sqrt{t^2 - \frac{z^2}{\varsigma^2}} = \zeta \quad , \quad \sqrt{\left(\frac{\zeta}{2}\right)^2 + \frac{z^2}{\varsigma^2}} = \sqrt{\frac{t^2 - \frac{z^2}{\varsigma^2}}{4} + \frac{z^2}{\varsigma^2}}$$

$\vec{t} + \vec{z} = \vec{u}$ **, id est ,**

$$u = 2\sqrt{\frac{t^2 + 3\frac{z^2}{\varsigma^2}}{4}} = \sqrt{t^2 + 3\frac{z^2}{\varsigma^2}} = \varsigma^{-1}\sqrt{x^2 + y^2 + 4z^2}$$

the coordinates of u **are** $\frac{x}{\varsigma}i \ , \ \frac{y}{\varsigma}j \ , \ 2\frac{z}{\varsigma}k$

$\vec{u} + \vec{x} = \vec{v}$ **, id est ,**

$$v = \sqrt{t^2 + 3\frac{x^2}{\varsigma^2} + 3\frac{z^2}{\varsigma^2}} = \varsigma^{-1}\sqrt{4x^2 + y^2 + 4z^2}$$

the coordinates of V **are** $2\frac{x}{\varsigma}i \ , \ \frac{y}{\varsigma}j \ , \ 2\frac{z}{\varsigma}k$

$\vec{v} + \vec{y} = \vec{w}$ **, id est ,**

$$w = \sqrt{t^2 + 3\frac{x^2}{\varsigma^2} + 3\frac{y^2}{\varsigma^2} + 3\frac{z^2}{\varsigma^2}} = \frac{2}{\varsigma}\sqrt{x^2 + y^2 + z^2} = 2t = \frac{\mathfrak{z}}{\varsigma}$$

the coordinates of the Position vector W **are** $2\frac{x}{\varsigma}i \ , \ 2\frac{y}{\varsigma}j \ , \ 2\frac{z}{\varsigma}k$

Obviously, a Variation of ς **or** t **or both implies the correlated variation of the length of the Space Axes and therefore of the Position axis**

SPACETIME ROTATIONS

Now, we know the length of the spacetime quaternion $\mathfrak{z} = w\varsigma$ **and the vector space consisting of 9 vectors:**

$$\overrightarrow{0x} \;,\; \overrightarrow{0y} \;,\; \overrightarrow{0z} \;,\; t \;,\; u \;,\; v \;,\; w \;,\; \varsigma \;,\; \overrightarrow{0\mathfrak{z}}$$

where all the vectors are linearly dependent.

The Space and Time and Speed in the Quaternion Vector Space are Linearly Dependent.

The SO(3) Group Spacetime Rotation is $\mathfrak{z} w \mathfrak{z}^*$ **where** w **is the position vector and** $\mathfrak{z}^*$ **is the conjugate of the STQ**

and to obtain the quaternion matrices:

□	i	j	k
i	-1	k	$-$j
j	$-$k	-1	i
k	j	$-$i	-1

$$\begin{matrix} \vec{x}\vec{x} & \vec{x}\vec{y} & \vec{x}\vec{z} \\ \vec{x}\vec{y} & \vec{y}\vec{y} & \vec{y}\vec{z} \\ \vec{x}\vec{z} & \vec{y}\vec{z} & \vec{z}\vec{z} \end{matrix} \quad \rightarrow \quad \begin{bmatrix} i & i & i \\ j & j & j \\ k & k & k \end{bmatrix} \cdot \begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} -1 & k & -j \\ -k & -1 & i \\ j & -i & -1 \end{bmatrix}$$

Then by the Euler's formula, we have

$$e^{\frac{\theta}{2}(xi+yj+zk)} = \cos\frac{\theta}{2} + (xi + yj + zk)\sin\frac{\theta}{2}$$

where θ **is the rotation angle, we divide it by 2 to represent rotations in both positive and negative directions.**

Thus:

$$\mathfrak{z} = \begin{matrix} -\cos\frac{\theta}{2} & 2zk\ \sin\frac{\theta}{2} & -2yj\ \sin\frac{\theta}{2} \\ -2zk\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & 2xi\ \sin\frac{\theta}{2} \\ 2yj\ \sin\frac{\theta}{2} & -2xi\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{matrix}$$

$$w = \begin{matrix} -\cos\frac{\theta}{2} & 2\frac{z}{\varsigma}k\ \sin\frac{\theta}{2} & -2\frac{y}{\varsigma}j\ \sin\frac{\theta}{2} \\ -2\frac{z}{\varsigma}k\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & 2\frac{x}{\varsigma}i\ \sin\frac{\theta}{2} \\ 2\frac{y}{\varsigma}j\ \sin\frac{\theta}{2} & -2\frac{x}{\varsigma}i\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{matrix}$$

$$
\mathfrak{z}^* = \begin{matrix} -\cos\frac{\theta}{2} & -2zk\sin\frac{\theta}{2} & 2yj\sin\frac{\theta}{2} \\ 2zk\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & -2xi\sin\frac{\theta}{2} \\ -2yj\sin\frac{\theta}{2} & 2xi\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{matrix}
$$

$$
\mathfrak{z}w\mathfrak{z}^* =
$$

$$
\begin{matrix} -\frac{4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}}{\varsigma}\left(y^2+z^2\right)-\cos^3\frac{\theta}{2} & 4z^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} & 4y^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} \\ 4z^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} & -\frac{4\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}}{\varsigma}\left(x^2+z^2\right)-\cos^3\frac{\theta}{2} & 4x^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} \\ 4y^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} & 4x^2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(1+\frac{1}{\varsigma}\right)-\frac{8xyz\sin^3\frac{\theta}{2}}{\varsigma} & -\frac{4\sin\frac{\theta}{2}^2\cos\frac{\theta}{2}}{\varsigma}\left(x^2+y^2\right)-\cos^3\frac{\theta}{2} \end{matrix}
$$