Q_8 *SpaceTime Rotations*

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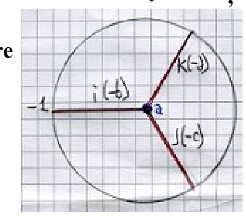
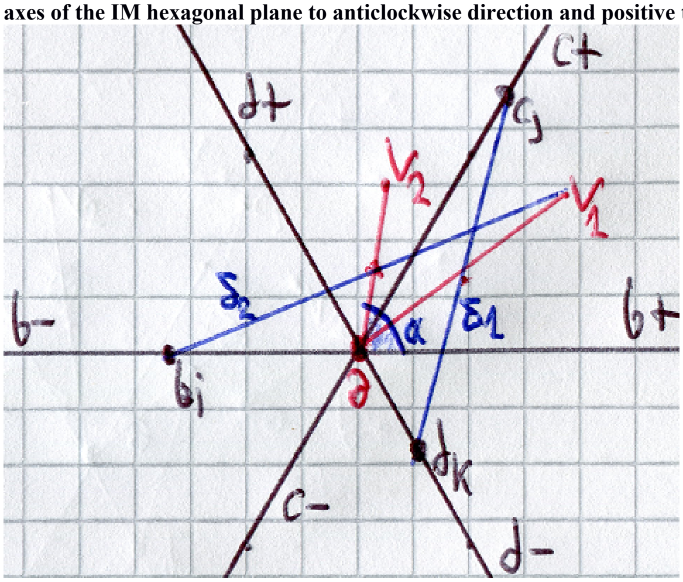
Given the following relation beetween the conjugate cube roots of unity and the quaternions: $z^* = e^{\frac{(i+j+k)\pi(2k+1)}{3}} \mid k \in N = \sqrt[3]{-1}$ **,**

hence $z^{*3} = i^2 = j^2 = k^2 = ijk = -1$ **, here below You can see a 4-Dimension normed vector space, where a conjugate cube root of unity generates 3 semi-axes and the Quaternion reflect them and generates a Q8 group where** a **orthogonally added to the IM-plane where lie** $bi\ ,\ cj\ ,\ dk$ **is the real axis. Conventionally, we consider positive the first 3 axes of the IM hexagonal plane to anticlockwise direction and positive the real axis above the IM plane.**





$\vec{v}_2 = \vec{v}_1 \pm bi$ **where** $\vec{v}_1 = \pm cj \pm dk$ **is the vector generated on the IM-plane starting from the origin of the 4 axes.**

Then $\vec{v} = \pm a + \vec{v}_2$ **is the vector that defines the initial spatial rotation axis generated by 4 input values of a quaternion, one per axis.**

This method allows us to use NON-ZERO REAL PART in the construction of the rotation quaternions.

The orthonormal components of the normed vector $\vec{v}$ **are on Cartesian axes** $x\ ,\ y\ ,\ z \in R$ **where Z is the real part and X and Y include the 3 axes of the imaginary part**

Obtained the cartesian 3D orthogonal coordinates, we can directly refer them, since $x\ ,\ y\ ,\ z$ **also include and replace the 4 input values of the quaternion.**

Converting the hypercomplex number into real 3D space vector allows us to use the Quaternions in the SpaceTime Rotations and in the $R^{3,1}$ **Minkowski spacetime for several purposes**

Trigonometric Calculations:

$$\vec{v}_1 = 2\frac{\sqrt{2c_j^2 + 2d_k^2 - \sqrt{c_j^2 + d_k^2 - 2c_jd_k\cos\frac{2\pi}{3}}^{\,2}}}{2} = \sqrt{c_j^2 + d_k^2 + 2c_jd_k\cos\frac{2\pi}{3}}$$

$$\delta_1 = \sqrt{c_j^2 + d_k^2 - 2c_jd_k\cos\frac{2\pi}{3}}$$

$$\vec{v}_2 = \sqrt{\vec{v}_1^{\,2} + b_i^2 + 2\vec{v}_1 b_i \cos\left(\frac{2\pi}{3} + \arccos\left(\frac{\frac{\vec{v}_1^{\,2}}{4} + c_j^2 - \frac{\delta_1^2}{4}}{2\,\vec{v}_1\,c_j^2}\right)\right)}$$

$$\vec{v} = \sqrt{a^2 + \vec{v}_2^{\,2}}$$

<u>Cartesian Coordinates</u> of $\vec{v}$ **:**

$$x = \vec{v}_2 \cos\alpha$$

$$y = \vec{v}_2 \sin\alpha$$

$$z = a$$

Calculation of the Angle α **:**

$$\alpha = \vec{v}_2 \hat{\ } b_+ = b_+ \hat{\ } v_1 + v_2 \hat{\ } v_1 \text{ where}$$

$$b_+ \hat{\ } v_1 = \pi - \frac{2\pi}{3} - \arccos\left(\frac{\frac{\vec{v}_1^{\,2}}{4} + c_j^2 - \frac{\delta_1^2}{4}}{2\,\vec{v}_1\, c_j^2}\right)$$

$$v_2 \hat{\ } v_1 = \arccos\left(\frac{\frac{\vec{v}_2^{\,2}}{4} + v_1^2 - \frac{\delta_2^2}{4}}{\vec{v}_2\, \vec{v}_1}\right)$$

Thus a Space-Time vector $-t + x + y + z$ **includes a Quaternion, then:**

then $v_2 = \dfrac{x}{\cos\alpha} = \dfrac{y}{\sin\alpha} = \vec{x} + \vec{y}$

then $-ti + v_2 j + zk$ **is a new kind of quaternion: <u>the Spacetime Quaternion</u>**

then $p = -ti + v_2 j + zk$ **is the position vector**

then $\mathbf{q} = e^{\frac{\theta}{2}(-ti + v_2 j + zk)} = \cos\frac{\theta}{2} + (-ti + v_2 j + zk)\sin\frac{\theta}{2}$

then $\mathbf{q} = e^{-\frac{\theta}{2}(-ti + v_2 j + zk)} = \cos\frac{\theta}{2} + (ti - v_2 j - zk)\sin\frac{\theta}{2}$

where θ is the rotation angle

by Hamilton product, their SPACE-TIME ROTATION is qpq^* that gives a new position vector at the time t'

its Rotation Matrix is:

$$
\begin{matrix}
t^2(1-\cos\theta)+\cos\theta & -tv_2(1-\cos\theta)-z\sin\theta & -tz(1-\cos\theta)+v_2\sin\theta \\
-tv_2(1-\cos\theta)+z\sin\theta & v_2^2(1-\cos\theta)+\cos\theta & v_2 z(1-\cos\theta)+t\sin\theta \\
-tz(1-\cos\theta)-v_2\sin\theta & v_2 z(1-\cos\theta)-t\sin\theta & z^2(1-\cos\theta)+\cos\theta
\end{matrix}
$$

The Distance beetween Point $\mathbf{p}$ and Point $\mathbf{p'}$ is:

$$
\begin{matrix}
(t-t')^2(1-c)+c & -(t-t')(v_2-v_2')(1-c)-(z-z')s & -(t-t')(z-z')(1-c)+(v_2-v_2')s \\
-(t-t')(v_2-v_2')(1-c)+(z-z')s & (v_2-v_2')^2(1-c)+c & (v_2-v_2')(z-z')(1-c)+(t-t')s \\
-(t-t')(z-z')(1-c)-(v_2-v_2')s & (v_2-v_2')(z-z')(1-c)-(t-t')s & (z-z')^2(1-c)+c
\end{matrix}
$$

where $s = \sin\theta \;\;,\;\; c = \cos\theta$

We want to know the Rotation Velocity s to move from $p = -t + v_2 + z$ to $p' = -t' + v_2' + z'$.

We use the Space-time interval $-s^2(t+t')^2 + (x-x')^2 + (y-y')^2 + (z-z')^2 = 0$ as *speed·time = space vector*
$s^2(t+t')^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ and where we arbitrarily replaced the constant Speed of Light with the dependent variable s

$$
s = \left(\frac{(x-x')^2 + (y-y')^2(z-z')^2}{(t-t')^2}\right)^{0.5}
$$

known s, we can calculate the minkowski scalar product of two space-time vectors $-s^2tt' + xx' + yy' + zz' = \psi$